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Attitude Stabilization with Unknown Bounded Delay in Feedback Control Implementation

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This paper addresses the problem of stabilizing attitude dynamics with an unknown constant delay in feedback with a known strict upper bound. A novel modification to the concept of the complete-type Lyapunov–Krasovskii functional plays a crucial role toward ensuring stability robustness to time delay in the control design. The control law is linear in states, and the resulting closed-loop equations are partitioned to form a nominal system with a perturbation. After obtaining necessary and sufficient exponential stability conditions for the nominal system, a complete-type Lyapunov–Krasovskii functional is constructed. As an intermediate step, an analytical solution for the underlying Lyapunov matrix is obtained. A systematic numerical optimization process is employed here to choose various controller gain parameters so that the region of attraction estimate is maximized. The closed-loop dynamics are shown to be exponentially stable inside the region of attraction estimate. To the authors' best knowledge, this is the first result that provides stable closed-loop control design for the attitude dynamics problem with an unknown delay in feedback.

I. Introduction

THE problem of rigid body attitude dynamics and control has been studied extensively over the last few decades due to its significance with respect to a wide range of applications, ranging from rigid aircraft and spacecraft systems to coordinated robot manipulators [1]. For example, rigid spacecraft applications, in particular, require highly accurate pointing maneuvers. These performance requirements necessitate the spacecraft model to be essentially nonlinear so that large amplitude angle orientations are accurately stabilized. Several results exist on feedback with attitude dynamics tackling various aspects of the attitude control problem. For instance, it is well known that linear feedback of the states stabilizes the dynamics [1].

The problem of stabilization of attitude dynamics when feedback is time delayed is challenging to solve. Time delay would arise in feedback due to communication delays and/or processing delays. As a result, the underlying dynamics do not contain a time delay by themselves, but they are subjected to one in the closed loop since current information of states is not available for feedback. Various classical feedback linearization and/or Lyapunov-based control design techniques cannot be employed, since the feedback does not contain current values of states. This leads to analyzing the problem from a time-delay system (TDS) framework. To the best of ou rknowledge, there is just one result on the attitude control problem that addresses the presence of time delay in feedback [2]. The authors of [2] have proposed a velocity-independent time-delay controller for regulating the attitude orientation of a rigid body. Rodrigues parameters (RPs) are employed to represent the attitude orientation. The control design involves filter construction to avoid velocity measurement. Sufficient conditions for exponential stability of the system inside a region of attraction, for which the estimate is calculated, and a measure to evaluate the system rate of convergence of the system to a desired setpoint are presented. However, the control design of [2] requires the delay to be known precisely, which is a rather restrictive condition. Furthermore, the control implementation provides stability conditions for sufficiently small time delays. Moreover, the estimate of the region of attraction was found to be quite conservative. Our work will relax the restriction requiring precise knowledge of time delay and obtain improved estimates of the region of attraction. In this formulation, the time delay is not required to be sufficiently small in order to obtain region-of-attraction estimate conditions.

Stability analysis of nonlinear TDSs presents a lot more challenges when compared with that of nonlinear systems without delays, in general. The problem difficulty depends considerably on the nature of delay present in the system. A further layer of difficulty is when the delay itself is unknown. Lyapunov-based stability methods, which have proven to be a popular tool for control design in nonlinear systems, have been extended to TDS. Krasovskii proposed the idea of a Lyapunov functional [3] (i.e., $V(t, x_t)$, $x_t \in [x(t-\tau), x(t)]$, and $\tau > 0$), also called a Lyapunov–Krasovskii (L–K) functional, as opposed to a Lyapunov function $V(t, x)$ in order to obtain sufficient conditions for stability of certain classes of TDS. The problem of constructing a L–K functional for any given TDS is comparatively more difficult than constructing a Lyapunov function for its delay-free counterpart, as will be shown in later sections of this paper. Moreover, there is no constructive method to formulate a L–K functional for a particular TDS. Classical L–K methods cannot be applied to systems that are unstable in the absence of delayed terms; that is, the delayed term is treated as a perturbation causing instability to the delay-free nominal system. As a result, the problem of L–K functional construction, in general, has generated considerable interest in TDS research [4,5].

During the last few years, Kharitonov and Zhabko proposed a constructive method to formulate a complete-type (i.e., completely quadratic) L–K functional for any given TDS that was known to be exponentially stable [6]. Furthermore, this complete-type L–K functional was employed for robust stability analysis for bounded perturbations to the exponentially stable nominal system, and estimates on how large the drift term could be were obtained. Melchor-Aguilar and Niculescu further elaborated this idea as a method to achieve regional stabilization, when the perturbation could be nonlinear and bounded by a linear growth, and extended the analysis to systems with a time-varying delay [7]. An important distinction in the construction of the complete-type L–K functional is that no assumptions are imposed on the system in the absence of delayed terms. For example, consider the scalar integrator $\dot{x}(t) = u(t)$ with delayed feedback $u(t) = -bx(t-\tau)$. It is well known in literature

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[8] that this scalar integrator with delayed feedback is exponentially stable if, and only if, $0 < b < \pi/2\tau$. Assuming that this condition is met, a complete-type L–K functional can be constructed for this system and used for robustness analysis when a drift term $f[x(t)]$ is present, as in $\dot{x}(t) = -bx(t-\tau) + f[x(t)]$ [7]. The complete-type L–K functional has since been applied to a biological problem [9]. However, the estimate on the region of attraction obtained was found to be somewhat conservative [9]. Furthermore, the analysis has been extended to construct a complete-type L–K functional that had a cross term in the time derivative [10]. This generalization reduced the conservatism of the estimate. A notable requirement for this technique in calculating an estimate for the region of attraction of the given system is precise knowledge of the time delay, which can be restrictive.

Substantial efforts have focused on stability analysis of linear systems with uncertain delays [11–13]. Reference [11] proposes a noncomplete-type L–K functional, which is employed toward stability analysis of linear systems but cannot be used for nonlinear systems, since it does not have a known lower bound. A known lower bound is crucial toward obtaining a meaningful region-of-attraction estimate. Our modification to the complete-type L–K functional in [6,7] ensures that, for an unknown constant time delay τ with a known strict upper bound $\tau_{\max}$, it is possible to obtain a known lower bound on the complete-type L–K functional. References [12,13] both propose new complete-type L–K functionals, which are different from the complete-type L–K functional proposed in this work. There is an added perturbation term to the original complete-type L–K functional construction from [6], which disappears when the perturbation in the delay goes to zero. The derivative of the L–K functional has quadratic terms of the state as well as the state derivative. The construction in the aforementioned references enables reduction in conservatism when compared with [11]. However, [12,13] do not provide a known lower bound for their complete-type L–K functional. Moreover, complications in the robustness analysis result in a nonintuitive method in choosing the free parameters. The addition of an extra term containing the derivative complicates the analysis in order to determine the analogous Lyapunov matrix $U(\theta)$. There are specific cases, as mentioned by [12], where the Lyapunov matrix solution is inaccurate. This, in turn, brings in conservative results, as noted by [12], since the associated numerical optimization process would no longer be as straightforward as proposed in our method.

This paper considers regionally stabilizing attitude dynamics with unknown delay in the feedback. A strict upper bound on the time delay is known. We propose a novel modification to the concept of the complete-type L–K functional. This modification is made in order to include robustness to time delay in the control design. Modified RPs (MRPs) are employed to represent the attitude orientation, a choice that is crucial to the development stated in this paper. The closed-loop dynamics are separated into the form of nominal dynamics, with a perturbation or drift term. This separation of the dynamics enables us to employ the modified theory of complete-type L–K functional for robust stability analysis with respect to the perturbed dynamics and time delay. The perturbation term obtained is such that it is a function of the current value of the states alone. Numerical optimization is employed in order to choose parameters such that the region of attraction is maximized. To realistically simulate the TDS, the initial condition trajectory is generated by propagating the attitude dynamics equations without control action. Analysis ensures that the states do not leave the estimate of the region of attraction during the initial condition time interval. Simulations demonstrate the validity of the approach. This is, to the best of our knowledge, the first result that provides stable closed-loop control design for the attitude dynamics problem with an unknown delay in feedback.

The rest of the paper is organized as follows. Section II formulates the attitude stabilization problem with delay in feedback. The extension of complete-type L–K functionals to nonlinear systems with unknown constant time delay is presented in Sec. III. The application of the complete-type L–K method toward regional stabilization of attitude dynamics is addressed in Sec. IV. Section V describes simulation examples of the proposed control strategy, whereas the summary and discussion are presented in Sec. VI.

The following notation will be employed through the remainder of the paper. For any symmetric positive-definite matrix $Q \in \mathbb{R}^{n\times n}$, we have Q_m and Q_M as the smallest and largest eigenvalues of Q, respectively. For any state vector $x(t) \in \mathbb{R}^n$, we denote its time-delayed value by $x(t-\tau)$, where $\tau > 0$ is the time delay. The subscript d is used in representing the nominal system dynamics. The norm $\|.\|$ used is the two norm for vectors and the induced norm for matrices.

II. System Dynamics and Problem Statement

The problem considered in this work is that of stabilizing attitude dynamics of a rigid body with an unknown constant delay in feedback. We employ MRPs [14] in order to represent the attitude orientation of the rigid body. The system dynamics can be expressed as

$$\dot{\sigma}(t) = \tfrac{1}{4}B[\sigma(t)]\omega(t) \tag{1}$$

$$J\dot{\omega}(t) = -\omega^{\times}(t)J\omega(t) + u(t) \tag{2}$$

where $\sigma(t) \in \mathbb{R}^3$ denotes the MRP vector, $J = J^T \in \mathbb{R}^{3\times3}$ is the positive-definite mass inertia matrix, and $\omega(t) \in \mathbb{R}^3$ is the angular velocity prescribed in a body-fixed reference frame. For any a, $b \in \mathbb{R}^3$, the skew symmetric matrix operator $a^{\times}$ represents the cross product between a and b with $a^{\times}b = a \times b$, and $B(\sigma)$ is defined as

$$B(\sigma) = [(1-\sigma^T\sigma)I_3 + 2\sigma^{\times} + 2\sigma\sigma^T] \tag{3}$$

The time delay τ is an unknown constant τ with $0 \le \tau < \tau_{\max}$, with $\tau_{\max}$ being known. Initial condition trajectories for Eqs. (1) and (2) are generated by propagating the governing dynamics without control action over the initial condition time interval. Initial conditions $\sigma_0 \in \mathbb{R}^3$ and $\omega_0 \in \mathbb{R}^3$ lie within the region-of-attraction estimate (to be established in the sequel) and, moreover, state trajectories do not escape from the calculated estimate for the region of attraction during the control-free propagation phase. It is assumed that delayed state measurements alone (i.e., $\sigma(t-\tau)$ and $\omega(t-\tau)$) are available for feedback purposes. Equation (2) shows that the feedback contains state information at time instant $t-\tau$, where τ is the time delay present in the feedback and is assumed to be unknown. We assume perfect knowledge of $\tau_{\max}$, which is a strict upper bound on the feedback time delay. In the absence of delay (i.e., with $\tau = 0$), it is well documented that linear feedback of states $u(t) = -K_1\sigma(t) - K_2\omega(t)$, with K_1 and K_2 being arbitrary positive constants, asymptotically stabilizes the dynamics in Eqs. (1) and (2) [14]. However, if delay is present and not accounted for in the input, the result is increased closed-loop oscillations and even instability [2]. The control objective is to achieve stabilization of the states (i.e., to ensure that $\sigma(t) \to 0$ and $\omega(t) \to 0$) in the presence of unknown constant delay in feedback loop.

III. Complete-Type Lyapunov–Krasovskii Functional Approach

It is well known that there is considerable difficulty in constructing a L–K functional for stability analysis of any given system. To circumvent this obstacle, Kharitonov and Zhabko formulated a completely quadratic L–K functional for any linear TDS that was known to be exponentially stable [6]. The same functional was further adopted for robust stability analysis due to nonlinear perturbations and, depending on the properties of the perturbation term present, regional stabilization conditions were obtained [7]. The complete-type L–K technique can be thought of as an extension of robustness analysis using the so-called Lyapunov equation for finite dimensional systems. However, a major hypothesis so far has been that the actual time delay is assumed to be exactly determined, which can be a restrictive condition in applications. In the following development, we modify the L–K functional concept in order to include robustness to time delay as well while still preserving the original features of the complete-type L–K approach. Sufficient stability conditions enable us to obtain an estimate for the region of attraction for a class of nonlinear TDSs.

We consider the following class of nonlinear TDSs:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + f[x(t)]$$
$$x(\theta) = \varphi(\theta), \qquad \theta \in [-\tau, 0] \tag{4}$$

where τ is a positive unknown constant, such that $0 \le \tau < \tau_{\max}$, where $\tau_{\max}$ is perfectly known. The nonlinear system is partitioned to formulate a nominal linear system:

$$\dot{x}_d(t) = A_0 x_d(t) + A_1 x_d(t-\tau)$$
$$x_d(\theta) = \varphi_d(\theta), \qquad \theta \in [-\tau, 0] \tag{5}$$

with a nonlinear perturbation $f[x(t)]$. The term $A_1 x(t-\tau)$ can represent a contribution from delayed state feedback.

A. Nominal System

Assume that A_0 and A_1 are such that Eq. (5) is exponentially stable. This means that $\exists \mu \ge 1$ and $\alpha > 0$, such that

$$\|x(t)\| \le \mu \|\varphi\|_\tau e^{-\alpha t} \tag{6}$$

where $\|\varphi\|_\tau$ is given by

$$\|\varphi\|_\tau = \sup_{\theta \in [-\tau, 0]} \|\varphi(\theta)\| \tag{7}$$

To construct a complete-type L–K functional, we start with the observation that, given any symmetric positive-definite matrices $W_i \in \mathbb{R}^{n\times n}$ ($i = 0$, 1, and 2), it follows from employing the Leibnitz rule, substituting $\zeta = t + \theta$, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{-\tau}^{0} x^T(t+\theta)[W_1 + (\tau_{\max}+\theta)W_2]x(t+\theta)\,\mathrm{d}\theta\right\}$$
$$= x^T(t)(W_1 + \tau_{\max}W_2)x(t) - x^T(t-\tau)\tilde{W}_1 x(t-\tau)$$
$$- \int_{-\tau}^{0} x^T(t+\theta)W_2 x(t+\theta)\,\mathrm{d}\theta$$

where $\tilde{W}_1 \doteq W_1 + (\tau_{\max} - \tau)W_2$. So, if there exists a functional $V_0(x_t)$ such that $\forall\, t \ge 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}V_0(x_t) = -w_0(x_t) = -x^T(t)\tilde{W}x(t)$$

where $W \doteq [W_0 + W_1 + \tau_{\max}W_2]$, then the first time derivative of the functional

$$V(x_t) = V_0(x_t) + \int_{-\tau}^{0} x^T(t+\theta)[W_1 + (\tau_{\max}+\theta)W_2]x(t+\theta)\,\mathrm{d}\theta$$

is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x_t) = -x^T(t)W_0 x(t) - x^T(t-\tau)\tilde{W}_1 x(t-\tau)$$
$$- \int_{-\tau}^{0} x^T(t+\theta)W_2 x(t+\theta)\,\mathrm{d}\theta = -w(x_t) \tag{8}$$

It has been shown in [6] that, given any positive-definite $\tilde{W}$, a necessary and sufficient condition for the existence of $V_0(x_t)$ is the exponential stability of the nominal system represented by Eq. (5). In this case,$V_0(x_t)$ is given by

$$V_0(x_t) = \int_0^{\infty} x^T(t)\tilde{W}x(t)\,\mathrm{d}t$$

$V(x_t)$ is called the complete-type L–K functional associated with Eq. (5) and is of the form

$$V(x_t) = x^T(t)U(0)x(t) - 2x^T(t)\int_{-\tau}^{0} U(-\tau-\theta)A_1 x(t+\theta)\,\mathrm{d}\theta$$
$$+ \int_{-\tau}^{0}\int_{-\tau}^{0} x(t+\theta_1)A_1^T U(\theta_1-\theta_2)A_1 x(t+\theta_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2$$
$$+ \int_{-\tau}^{0} x^T(t+\theta)[W_1 + (\tau_{\max}+\theta)W_2]x(t+\theta)\,\mathrm{d}\theta \tag{9}$$

where $U(\theta)$ is called the Lyapunov matrix [6][‡] and is defined as

$$U(\theta) = \int_0^{\infty} K^T(t)\tilde{W}K(t+\theta)\,\mathrm{d}t \tag{10}$$

where $K(t)$ is the unique matrix function that satisfies[§]

$$\dot{K}(t) = A_0 K(t) + A_1 K(t-\tau)$$
$$K(\theta) = 0, \qquad \theta < 0; \qquad K(0) = I \tag{11}$$

The Lyapunov matrix is well defined because $K(t)$ vanishes for $t < 0$ and approaches zero exponentially as $t \to \infty$, since the nominal system in Eq. (5) is exponentially stable. The Lyapunov matrix $U(\theta)$ in Eq. (10) can be shown to satisfy the second-order matrix differential equation

$$U''(\theta) = U'(\theta)A_0 - A_0^T U'(\theta) + A_0^T U(\theta)A_0 - A_1^T U(\theta)A_1 \tag{12}$$

subjected the mixed boundary conditions

$$U'(0) + [U'(0)]^T = -\tilde{W} \tag{13}$$

$$U'(0) = U(0)A_0 + U^T(\tau)A_1 \tag{14}$$

Note here that we denote $U'(\theta) = \mathrm{d}U(\theta)/\mathrm{d}\theta$. Also, it follows from Eqs. (10–14) that the Lyapunov matrix $U(\theta)$ is symmetric at $\theta = 0$ [6]:

$$U(\theta) = U^T(-\theta) \tag{15}$$

Equation (12), together with boundary conditions from Eqs. (13) and (14) and the symmetry property from Eq. (15), will be employed in order to find an analytical solution for the Lyapunov matrix associated with the nominal system chosen to represent the attitude dynamics.

We establish upper and lower bounds on $V(x_t)$, which will be critical in order to obtain a region of attraction estimate through robust stability analysis of the nonlinear system represented by Eq. (4).

Lemma 1: Let the nominal system (5) be exponentially stable. For any positive-definite W_j ($j = 0$, 1, and 2), $V(x_t)$, defined in Eq. (9), has the following properties for some positive constants α_1, α_2, and α_3, where 1) $\alpha_1\|\varphi(0)\| \le V(x_t) \le \alpha_2\|\varphi\|_\tau^2$ and 2) $\dot{V}(x_t) \le -\alpha_3\|x(t)\|^2$.

Proof. Since the nominal system is exponentially stable, we have from Eq. (8)

$$\frac{\mathrm{d}V(x_t)}{\mathrm{d}t} = -w(x_t) \le -\alpha_3\|x(t)\|^2, \qquad t \ge 0 \tag{16}$$

where $\alpha_3 = W_{0m}$. To find and prove a lower bound for $V(\varphi)$, define the functional

$$V_{\alpha_1}(\varphi) = V(\varphi) - \alpha_1\varphi^T(0)\varphi(0) \tag{17}$$

Differentiating $V_{\alpha_1}(\varphi)$ along the nominal system trajectory

[‡]In [6,7,15], the Lyapunov matrix in Eq. (10) is defined with $W = W_0 + W_1 + \tau W_2$, whereas our modification removes the restriction of precise knowledge of τ.

[§]$K(t)$ and $U(\theta)$ can be viewed as extensions of the state transition matrix and solution of the Lyapunov equation, respectively, from the domain of finite dimensional systems.

$$\frac{\mathrm{d}V_{\alpha_1}(x_t)}{\mathrm{d}t} = -w(x_t) - 2\alpha_1 x^T[A_0x(t) + A_1x(t-\tau)]$$

We have the following inequality

$$-2\alpha_1 x(t)^T[A_0x(t) + A_1x(t-\tau)] \le 2\alpha_1\|A_0\|\|x(t)\|^2 + \alpha_1\|A_1\|(\|x(t)\|^2 + \|x(t-\tau)\|^2)$$

This leads to

$$\frac{\mathrm{d}V_{\alpha_1}(x_t)}{\mathrm{d}t} \le -w_{\alpha_1}(x_t) \tag{18}$$

where

$$w_{\alpha_1}(\varphi) = [W_{0m} - \alpha_1(2\|A_0\| + \|A_1\|)]\|\varphi(0)\|^2 + [W_{1m} - \alpha_1\|A_1\|]\|\varphi(-\tau)\|^2$$

We choose α_1 such that

$$\alpha_1 \le \min\left\{\frac{W_{0m}}{2\|A_0\| + \|A_1\|}, \frac{W_{1m}}{\|A_1\|}\right\} \tag{19}$$

so that $w_{\alpha_1}(\varphi) \ge 0$. Integrating the inequality in Eq. (18) from zero to ∞ leads to

$$V_{\alpha_1}(\varphi) \ge \int_0^\infty w_{\alpha_1}(x_t)\,\mathrm{d}t \ge 0$$

and subsequently

$$V(\varphi) \ge \alpha_1\|\varphi(0)\|^2 \tag{20}$$

Let $u_0 = \sup_{\theta\in[0,\tau_{\max}]}\|U(\theta)\|$. We consider the following inequalities in order to establish an upper bound:

$$\varphi^T(0)U(0)\varphi(0) \le u_0\|\varphi(0)\|^2 - 2\varphi^T(0)\int_{-\tau}^0 U(-\tau-\theta)A_1\varphi(\theta)\,\mathrm{d}\theta \le u_0\|A_1\|\left(\tau_{\max}\|\varphi(0)\|^2 + \int_{-\tau}^0\|\varphi(\theta)\|^2\,\mathrm{d}\theta\right)\int_{-\tau}^0\varphi(\theta_1)^TA_1^TU(\theta_1 - \theta_2)A_1\varphi(\theta_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \le u_0\tau_{\max}\|A_1\|^2\int_{-\tau}^0\|\varphi(\theta)\|^2\,\mathrm{d}\theta\int_{-\tau}^0\varphi(\theta)^T[W_1 + (\tau_{\max}+\theta)W_2]\varphi(\theta)\,\mathrm{d}\theta \le \|W_1 + \tau_{\max}W_2\|\int_{-\tau}^0\|\varphi(\theta)\|^2\,\mathrm{d}\theta$$

which lead to the following inequality:

$$V(\varphi) \le \alpha_2\|\varphi\|_\tau$$

where

$$\alpha_2 = u_0(1+\tau_{\max})\max\{(1+\tau_{\max}\|A_1\|), \|A_1\|(1+\tau_{\max}) + \|W_1 + \tau_{\max}W_2\|\} \tag{21}$$

Remark 1. The expression for α_1 can be made less conservative by having it depend on the minimum eigenvalue of $\tilde{W}_1$ [where $\tilde{W}_1 = W_1 + (\tau_{\max} - \tau)W_2$] rather than the minimum eigenvalue of W_1, since$W_{1m} \le \tilde{W}_{1m}$ through the spectral shift property. However, $\tilde{W}_{1m}$ requires precise knowledge of the unknown time delay τ, which cannot be useful in characterizing an estimate for the α_1 parameter.

B. Analytical Solution to Lyapunov Matrix Ordinary Differential Equation

We apply Kronecker algebra [16] to the second-order linear matrix ordinary differential equation represented by Eq. (12) in order to obtain an analytical solution to the Lyapunov matrix $U(\theta)$. The differential equation can be written as a linear cascade system with $U_1 \doteq U$ and $U_2 \doteq U'$ as

$$U_1' = U_2 \tag{22}$$

$$U_2' = U_2A_0 - A_0^TU_2 + A_0^TU_1A_0 - A_1^TU_1A_1 \tag{23}$$

where A_0 and A_1 are such that the nominal system in Eq. (5) is exponentially stable. Define $Z = [\,U_1^T \quad U_2^T\,]^T$. We can rewrite Eqs. (22) and (23) as

$$Z' = P_0ZQ_0 + P_1ZQ_1 + MZ \tag{24}$$

where

$$P_0 = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ A_0^T & I_{n\times n}\end{bmatrix}, \qquad P_1 = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ A_1^T & 0_{n\times n}\end{bmatrix},$$

$$M = \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ 0_{n\times n} & -A_0^T\end{bmatrix}, \qquad Q_0 = A_0, \qquad Q_1 = -A_1$$

We define the stack vector X^S for any $X \in R^{m\times n}$ as

$$X^S = [\,x_{11} \quad x_{21} \quad \ldots \quad x_{n1} \quad x_{12} \quad \ldots \quad x_{1n} \quad \ldots \quad x_{mn}\,]^T \tag{25}$$

where x_{ij} ($i = 1, \ldots, m$ and $j = 1, \ldots, n$) are the elements of X. From the property of Kronecker products [12], we have

$$(PXQ)^S = (Q^{\mathrm{T}} \otimes P)X^S \tag{26}$$

Using Eq. (26) in order to use the stack transformation for Eq. (24) leads to

$$Z'(\theta)^S = (Q_0^T \otimes P_0 + Q_1^T \otimes P_1 + I_{2n\times 2n} \otimes M)Z(\theta)^S \tag{27}$$

since the nominal system is exponentially stable, and hence admits a unique solution for the Lyapunov matrix [6]. The general solution for Eq. (27) can be written as

$$Z(\theta)^S = e^{K\theta}Z(0)^S \tag{28}$$

where

$$K \doteq Q_0^T \otimes P_0 + Q_1^T \otimes P_1 + I_{2n\times 2n} \otimes M$$

The mixed boundary conditions in Eqs. (13) and (14) can be rewritten in terms of U_1 and U_2 as

$$U_2(0) + U_2^T(0) = -\tilde{W} \tag{29}$$

$$U_2(0) = U_1(0)A_0 + U_1^T(\tau)A_1 \tag{30}$$

To find the particular solution, we need to solve the boundary conditions from Eqs. (29) and (30). Employing the vector transformation again, we have

$$U_2(0)^S + EU_2(0)^S = -W^S \tag{31}$$

$$U_2(0)^S = (A_0^T \otimes I_{n\times n})U_1(0)^S + (A_1^T \otimes I_{n\times n})EU_1(\tau)^S \tag{32}$$

where E is the permutation matrix [16], which enables us to find the vector transformation for a transposed matrix X:

$$X^{T^S} = EX^S$$

Since $Z = [\,U_1^T \quad U_2^T\,]^T$, $U_1(\tau)$ can be expressed in terms of $U_1(0)$ and $U_2(0)$ by substituting $\theta = \tau$ in Eq. (28). As a result of the substitution, Eqs. (31) and (32) have $2n^2$ unknowns: namely, the elements of $U_1(0)$ and $U_2(0)$. The $(n-1)!$ unknowns and equations are redundant, since $U_1(0) = U(0)$ is symmetric from Eq. (15) and $U_2(0) + U_2^T(0)$ and $\tilde{W}$ are both symmetric. This leads to $2n^2 - (n-1)!$ equations with $2n^2 - (n-1)!$ unknowns, which has a unique solution. Since A_0 and A_1 are such that the nominal system is exponentially stable, the differential equation for the Lyapunov matrix $U(\theta)$ admits a unique solution for $\theta \ge 0$. The analytical solution for the Lyapunov matrix will be employed in order to obtain a supremum for $\|U(\theta)\|$ over the interval $\theta \in [0, \tau_{\max}]$, to be used in the robustness analysis of the nonlinear system in Eq. (4).

C. Robustness Analysis

The complete-type L–K functional formulated in Eq. (9) for the nominal system (5) is employed to calculate an estimate for the region of attraction for the nonlinear TDS represented by Eq. (4), with $f[x(t)]$ satisfying a Lipschitz condition in a certain vicinity of the origin

$$\|f[x(t)]\| < \gamma\|x(t)\| \tag{33}$$

where γ and δ are some positive constants. The following theorem extends results from [7] in order to include robustness to the unknown constant time delay.

Theorem 2: Let the nominal system [see Eq. (5)] be exponentially stable for all $\tau \in [0, \tau_{\max})$. Then, the nonlinear perturbed system [see Eq. (4)] is exponentially stable for all $\tau \in [0, \tau_{\max})$ inside the set $\|\varphi\| < \alpha_1/\alpha_2\delta$, where the drift term $f[x(t)]$ obeys the Lipschitz condition in Eq. (33), where

$$0 < \gamma < \min\left\{\frac{W_{0m}}{u_0(2 + \|A_1\|\tau_{\max})}, \frac{W_{2m}}{u_0\|A_1\|}\right\} \tag{34}$$

if

$$\|x(t)\| < \delta$$

$$u_0 = \sup_{\theta\in[0,\tau_{\max}]}\|U(\theta)\|$$

Proof. The derivative of Eq. (9) along the trajectories of the nonlinear system (4) is

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x_t) = -w(x_t) + 2f^T[x(t)]\left[U(0)x(t) - \int_{-\tau}^{0} U(-\tau-\theta)A_1x(t+\theta)\,\mathrm{d}\theta\right]$$

where $-w(x_t)$ is the derivative of the functional along the trajectories of the nominal system [see Eq. (8)]. From the Lipschitz condition, we have

$$\|f[x(t)]\| < \gamma\|x(t)\|$$

if

$$\|x(t)\| < \delta$$

It is easy to see that the following inequality holds

$$\|U(0)x(t) - \int_{-\tau}^{0} U(-\tau-\theta)A_1x(t+\theta)\,\mathrm{d}\theta\| \le u_0\left[\|x(t)\| + \|A_1\|\int_{-\tau}^{0}\|x(t+\theta)\|\,\mathrm{d}\theta\right]$$

It then follows that

$$2f^T[(x(t)]\left[U(0)x(t) - \int_{-\tau}^{0} U(-\tau-\theta)A_1x(t+\theta)\,\mathrm{d}\theta\right] \le \gamma u_0\left[(2 + \|A_1\|\tau_{\max})\|x(t)\|^2 + \|A_1\|\int_{-\tau}^{0}\|x(t+\theta)\|^2\,\mathrm{d}\theta\right]$$

Considering this inequality is the first time derivative of the complete-type L–K functional $V(x_t)$, we arrive at the following inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x_t) \le -[W_{0m} - \gamma u_0(2 + \|A_1\|\tau_{\max})]\|x(t)\|^2 - [W_{2m} - \gamma u_0\|A_1\|]\int_{-\tau}^{0}\|x(t+\theta)\|^2\,\mathrm{d}\theta \tag{35}$$

If γ satisfies Eq. (34), which happens if $\|x(t)\| < \delta$, then $\dot V(x_t)$ is negative definite for all trajectories inside the set determined by Eq. (33). To ensure that the Lipschitz condition is satisfied with $\dot V(x_t) < 0$, we need to show that $\|x(t)\| < \delta$ is an invariant set, since $\dot V(x_t) < 0$, and from Lemma 1, we obtain

$$\alpha_1\|x(t)\|^2 \le V(x_t) \le V(\varphi) \le \alpha_2\|\varphi\|_\tau^2, \qquad t \ge 0$$

which leads to

$$\|x(t)\| \le \sqrt{\frac{\alpha_2}{\alpha_1}}\|\varphi\|, \qquad t \ge 0 \tag{36}$$

Hence, in order for $\|x(t)\| < \delta$, we need

$$\|\varphi\|_\tau < \sqrt{\frac{\alpha_1}{\alpha_2}}\delta \tag{37}$$

From Eq. (35), $\dot V(x_t)$ can be written further as

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x_t) \le -\beta\left\{\|x(t)\|^2 + \int_{-\tau}^{0}\|x(t+\theta)\|^2\,\mathrm{d}\theta\right\}$$

where

$$\beta \doteq \min\{W_{0m} - \gamma u_0(2 + \|A_1\|\tau_{\max}), W_{2m} - \gamma u_0\|A_1\|\}$$

Furthermore, we can rewrite the upper bound on $V(x_t)$ as

$$V(t, x_t) \le \kappa\left\{\|x(t)\|^2 + \int_{-\tau}^{0}\|x(t+\theta)\|^2\,\mathrm{d}\theta\right\}$$

where

$$\kappa \doteq u_0\max\{(1 + \tau_{\max}\|A_1\|), \|A_1\|(1 + \tau_{\max}) + \|W_1 + \tau_{\max}W_2\|\}$$

With these upper bounds on $V(x_t)$, and its time derivative in place, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x_t) \le -\frac{\beta}{\kappa}V(x_t)$$

Using the comparison principle lemma [17], we have

$$\alpha_1\|x(t)\|^2 \le V(x_t) \le V(\varphi)e^{-(\beta/\kappa)t} \le \alpha_2\|\varphi\|_\tau^2 e^{(\beta/\kappa)t}, \qquad t \ge 0 \tag{38}$$

which leads to

$$\|x(t)\| \le \sqrt{\frac{\alpha_2}{\alpha_1}}\|\varphi\|_\tau e^{-(\beta/2\kappa)t}, \qquad t \ge 0 \tag{39}$$

and, consequently, the trivial solution $x(t) = 0$ of the nonlinear system in Eq. (4) is exponentially stable for all $\tau \in [0, \tau_{\max})$.

Remark 2: In [7], it is required that matrices W_j ($j = 0, 1$, and 2) be included in the calculation for γ through their minimum eigenvalues, as well through the expression for the supremum of the Lyapunov matrix u_0. However, the appearance of the minimum eigenvalue of W_1 is necessary only if the perturbation term depends on the delayed value of the states. In our formulation for the attitude dynamics problem, the perturbation turns out to be a function of the current value of the state alone (see Sec. IV), which permits specializations in order to reduce conservatism in region of attraction estimate.

Remark 3: Because of the modifications made in the construction of the complete-type L–K functional from Eq. (9), as well as the accompanying robustness analysis with its time derivative along with the Lyapunov matrix evaluation, calculating the size of the estimate of the region of attraction γ requires knowledge of a strict upper bound $\tau_{\max}$ of the time delay rather than its precise value τ. The actual delay τ is, however, present in the complete-type L–K functional, which is employed for analysis purposes alone.

Remark 4: In the finite dimensional case, we can calculate an estimate of a region for a corresponding finite dimensional nonlinear system by robustness analysis using the Lyapunov equation. In this case, the size of the estimate can be maximized through choice of parameters subject to the Lyapunov equation being satisfied. In the time-delay case, it is not so straightforward to maximize the size of the estimate γ, mainly due to an increase in the number of parameters and the accompanying constraints. However, it is possible to employ numerical optimization in a computing software such as MATLAB in order to maximize γ. This optimization will be performed after application to the attitude stabilization problem in Sec. IV.

IV. Attitude Stabilization with Unknown Constant Delay in Feedback

We apply the theoretical development in Sec. III toward the attitude stabilization problem with unknown constant delay. We formulate the nominal system and obtain an estimate of the region of attraction from the perturbed system formulation and the analysis of state norms over the initial condition interval.

A. Nominal System

The nominal system for applying this method to attitude dynamics is taken to be a block of three double integrators. We can rewrite the attitude dynamics from Eqs. (1) and (2) as

$$\dot{\sigma}(t) = \frac{\omega(t)}{4} + \frac{1}{4}\{B[\sigma(t)] - I_{3\times 3}\}\omega(t) \tag{40}$$

$$\dot{\omega}(t) = -J^{-1}\omega^{\times}(t)J\omega(t) + J^{-1}u(t) \tag{41}$$

Adding and subtracting $\omega/4$ to the attitude kinematics enables the construction of a perturbation term that satisfies the Lipschitz condition [see Eq. (33)]. The motivation behind adding and subtracting $\omega/4$ is to obtain a nominal system in the form of three double integrators. The residual term in the attitude kinematics, $[B(\sigma) - I_{3\times 3}]\omega/4$, is nonlinear.

At this point, we wish to mention an important reason for choosing the MRPs to represent the attitude kinematics rather than the more traditional (and globally nonsingular) quaternion parametrization. The kinematics equation with quaternion notation is expressed as [18]

$$\begin{bmatrix} \dot{\epsilon}(t) \\ \dot{\epsilon}_0(t) \end{bmatrix} = \begin{bmatrix} T(\epsilon(t)) \\ -\epsilon^T(t) \end{bmatrix}\omega \tag{42}$$

where $\epsilon(t) \in \mathbb{R}^3$ is the vector part of the quaternion and $\epsilon_0(t) \in \mathbb{R}$ is the scalar part of the quaternion with the norm constraint $\|\epsilon(t)\|^2 + |\epsilon(t)|^2 = 1$:

$$T[\epsilon(t)] = [\epsilon^{\times}(t) + \sqrt{1 - \epsilon^T(t)\epsilon(t)}I_{3\times 3}]/2$$

$T[\epsilon(t)]$ cannot be made homogenous in the position state ϵ by adding and subtracting some $\lambda\omega(t)$ (where $\lambda \in \mathbb{R}$), as was done with the MRP representation in Eq. (40) and, consequently, will not lead to definition of a drift term satisfying the Lipschitz-like condition in Eq. (33). This obstacle can be avoided by using the MRP representation.

We write the nominal system as

$$\dot{\sigma}_d(t) = \frac{\omega_d(t)}{4} \tag{43}$$

$$\dot{\omega}_d(t) = J^{-1}u(t) \tag{44}$$

Employing the state transformation $q = \omega/4$ and choosing the control

$$u(t) = -4J[\omega_n^2\sigma_d(t-\tau) + 2\xi\omega_n q_d(t-\tau)]$$

(with $\omega_n > 0$ and $\xi > 0$), which leads to the nominal system being a block of three decoupled double integrators with delayed feedback, as in

$$\dot{\sigma}_d(t) = q_d(t) \tag{45}$$

$$\dot{q}_d(t) = -\omega_n^2\sigma_d(t-\tau) - 2\xi\omega_n q_d(t-\tau) \tag{46}$$

On comparison with the generic nominal system [see Eq. (5)], we have the following expressions for A_0 and A_1:

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 0 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix}$$

The nonlinear perturbation is written as

$$F(\sigma, q) = \begin{bmatrix} [B(\sigma) - I_{3\times 3}]q \\ -4J^{-1}q^{\times}Jq \end{bmatrix} \tag{47}$$

From Eq. (47), we observe that the nonlinear perturbation is a function of the current value of the states alone. We analyze the characteristic equation for the double-integrator system in Eqs. (45) and (46) in order to determine the range of parameter values for which the nominal system is stable. Consider the double integrator with delayed feedback:

The stability of the nominal system is completely determined by its transcendental characteristic equation [19]:

$$s^2 + \omega_n^2 e^{-\tau s} + 2\xi\omega_n s e^{-\tau s} = 0 \tag{48}$$

Specifically, the system is exponentially stable if, and only if, the characteristic equation has no zero, or root, in the closed right half-plane. To determine the maximum value of delay the system can tolerate for given control parameters ω_n and ξ, it suffices to determine the critical values of the delay for which the roots of the characteristic equation move from the closed left half-plane to the imaginary axis, thus rendering the system unstable [8]. Thus, we wish to find the smallest deviation of the delay from zero (say, $\tau_{\max}$) such that the characteristic equation has imaginary roots; that is,

$$(j\omega)^2 + \omega_n^2 e^{-\tau_{\max} j\omega} + 2j\xi\omega_n\omega e^{-\tau_{\max} j\omega} = 0$$

where $j = \sqrt{-1}$. This leads to

$$-\omega^2 + (\omega_n^2 + 2j\xi\omega_n\omega)e^{-j\tau_{\max}\omega} = 0$$

Separating the real and imaginary parts leads to

$$-\omega^2 + \omega_n^2\cos(\tau_{\max}\omega) + 2\xi\omega_n\omega\sin(\tau_{\max}\omega) = 0$$
$$2\xi\omega_n\omega\cos(\tau_{\max}\omega) - \omega_n^2\sin(\tau_{\max}\omega) = 0$$

Combining the preceding two equations leads to $\cos(\tau_{\max}\omega) = \omega_n^2/\omega^2$ and $\sin(\tau_{\max}\omega) = 2\xi\omega_n/\omega$. With some standard algebraic manipulations, an analytical solution for $\tau_{\max}$ for a given ω_n^c and ξ^c is obtained as

$$\tau_{\max} = \frac{1}{\omega_n^c f}\sin^{-1}\frac{2\xi^c}{f}; \qquad f = \sqrt{2\xi^{c2} + \sqrt{1 + 4\xi^{c4}}} \tag{49}$$

Equation (49) enables us to obtain a maximum delay $\tau_{\max}$ for given ξ^c and ω_n^c. Equation (49) is a necessary and sufficient condition; that is, the system is critically stable $\tau = \tau_{\max}$ and unstable for $\tau > \tau_{\max}$. In another context, for a given $\tau_{\max}$, we can calculate a set of values that ξ and ω_n can take so that the system is exponentially stable. Reducing ω_n^c increases $\tau_{\max}$ for a constant ξ^c. Hence, for a given $\tau_{\max}$, any $\omega_n \in (0, \omega_n^c]$ results in an exponentially stable system $\forall\, \tau < \tau_{\max}$. Next, reducing ξ^c for a given ω_n^c results in a higher $\tau_{\max}$, since the term

$$\sin^{-1}\left(2\xi/\sqrt{2\xi^2 + \sqrt{1 + 4\xi^4}}\right)$$

is monotonic with respect to ξ. Again, the system is exponential for any $\xi \in (0, \xi^c]$, such that $\tau < \tau_{\max}$. In conclusion, the system is exponentially stable for any $\xi \in (0, \xi^c]$ and $\omega_n \in (0, \omega_n^c]$, such that $\tau < \tau_{\max}$. For example, Fig. 1 shows the ω_n^c-versus-ξ^c curve for three different values of $\tau_{\max}$: $\tau_{\max} = 0.2$, $\tau_{\max} = 0.5$, and $\tau_{\max} = 1$.

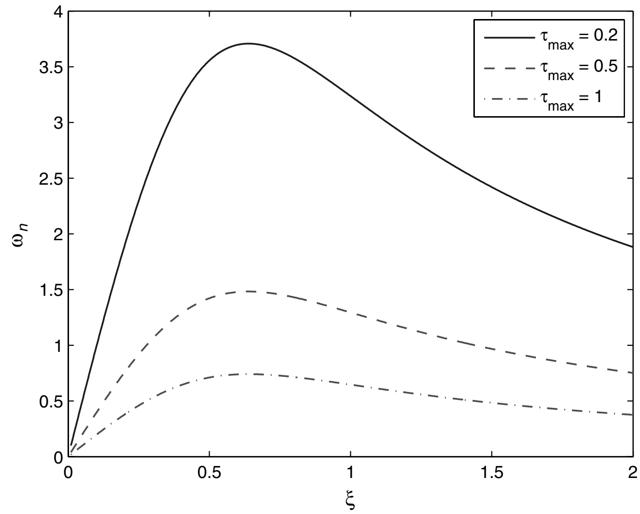

Fig. 1 ω_n^c **vs** ξ^c **for** $\tau_{\max} = 0.2$, $\tau_{\max} = 0.5$, **and** $\tau_{\max} = 1$.

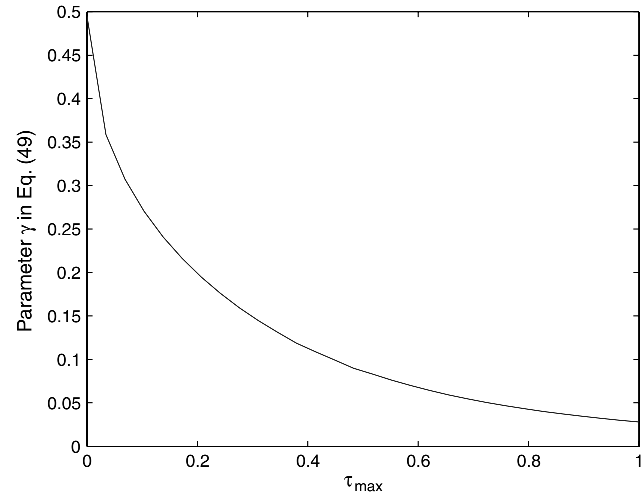

Fig. 2 γ **vs** $\tau_{\max}$ **variation.**

Figure 1 shows that, as $\tau_{\max}$ increases, the parameter space for permissible control gains ω_n and ξ shrinks.

Remark 5: The D-subdivision technique, as mentioned in [4], can be used as a method to obtain $\tau_{\max}$. This method maintains the time delay to be constant and finds regions in the parameter space inside the first such transition boundary for which stability holds. The delay continuity-based method used in this work finds the first such boundary, at which we have imaginary axis crossing of the characteristic roots [8,15], while keeping the controller parameters constant ω and ξ. This method, along with the fact that the expression for the $\tau_{\max}$ is monotonous with respect to the control parameters, is analogous to the D-subdivision technique and gives the same result for $\tau_{\max}$.

For stability analysis of the nonlinear system, we enforce the region-of-attraction condition on the states by obtaining γ from Eq. (34):

$$0 < \gamma < \min\left\{\frac{W_{0m}}{u_0(2 + \|A_1\|\tau_{\max})}, \frac{W_{2m}}{u_0\|A_1\|}\right\} \tag{50}$$

where A_0 and A_1 are given by Eq. (47). The analysis to obtain γ does not require knowledge of the structure of the perturbation term added to the nominal system from Eqs. (45) and (46), since it involves parameters associated with the nominal system and the Lyapunov matrix differential equation only. We use direct numerical optimization, choosing parameters ω_n, ξ, W_0, W_1, and W_2, such that γ is maximized, while keeping the nominal system exponentially stable. The nominal system is exponentially stable for the range of gain values determined by Eq. (49), and therefore admits a unique solution for the Lyapunov matrix.

We show the dependence of γ on $\tau_{\max}$ in Fig. 2, which does not need information from the nonlinear perturbation. As is expected, γ is observed to decrease as $\tau_{\max}$ increases. Figure 2 shows that the size of the perturbation term decreases as $\tau_{\max}$ increases. Figure 2 also shows that, as $\tau_{\max}$ approaches 0, γ approaches 0.5. To qualitatively interpret this observation, consider the expression for γ as obtained from Eq. (50). The Lyapunov matrix $U(\theta)$ is independent of W_2 at $\tau_{\max} = 0$, since $\tilde{W} = W_0 + W_1$ and the boundary conditions for the Lyapunov matrix collapse into an algebraic Lyapunov equation to be solved for $U(0)$ as

$$A^T U(0) + U(0)A = -\tilde{W} \tag{51}$$

where

$$A = A_0 + A_1$$

Subsequently, γ can be made independent of the second term from Eq. (50); that is,

$$\gamma = \frac{W_{0m}}{2u_0} \tag{52}$$

To maximize γ, W_1 needs to be negligible compared with W_0 (i.e., $W_0 \gg W_1$), and the resulting expression from Eq. (52) with $W_0 = I_{2\times 2}$ would be the famous Patel–Toda result [20], where $u_0 = U(0)$. By letting $\tilde{W} = I_{2\times 2}$, if we analytically solve for $U(0)$ in terms of control gains ω_n and ξ from Eq. (51), and maximize u_0, we obtain $u_0^* = 1$ with $\omega_n^* = 1.1731$ and $\xi^* = 0.5077$.

B. Analysis over Torque-Free Interval and Regional Stabilization

To realistically simulate the system, we require that there be no control over the initial condition time interval; that is, $t \in [-\tau, 0]$. It is highly important to ensure that, during this time evolution, the states do not escape from the estimated domain of attraction. This situation is tackled by calculating the upper bound on the states during this interval. Rewriting the system dynamics from Eqs. (1) and (2) with no control,

$$\dot{\sigma}(t) = \tfrac{1}{4}B[\sigma(t)]\omega(t) \tag{53}$$

$$J\dot{\omega}(t) = -\omega^\times(t)J\omega(t) \tag{54}$$

We calculate an upper bound for the angular velocity norm by employing the following positive-definite scalar function $V_\omega = \omega^T J\omega$. The time derivative of V_ω over the torque-free trajectory is zero. Hence, $J_m\|\omega\|^2 \le J_M\|\omega_0\|^2$. Define $\Lambda = \sqrt{J_M/J_m}$. This leads to an upper bound for the $\|\omega(t)\|$ over the initial condition interval as

$$\|\omega(t)\| \le \Lambda\|\omega_0\|, \qquad t \in [-\tau, 0] \tag{55}$$

An upper bound for $\|\sigma(t)\|$ over the initial condition interval is obtained by

$$\frac{\mathrm{d}\|\sigma(t)\|}{\mathrm{d}t} \le \|\dot{\sigma}(t)\| \le \frac{1}{4}\|B(\sigma(t))\|\|\omega(t)\|$$

From [14], $\|B(\sigma)\| = 1 + \sigma^2$, such that $\|\sigma\|$, being bounded and substituting for $\|\omega(t)\|$ from Eq. (55), leads to

$$\frac{4\dot{\sigma}(t)}{1 + \sigma^2(t)} \le \Lambda\|\omega_0\|$$

Integrating both sides from $-\tau$ to $t \in [-\tau, 0]$ and using the comparison principle lemma [17] leads to

$$\tan^{-1}[\sigma(t)] - \tan^{-1}(\sigma_0) \le \frac{\Lambda\|\omega_0\|(t + \tau)}{4}, \qquad t \in [-\tau, 0] \tag{56}$$

Note that Eq. (56) is satisfied if the following holds:

$$\|\sigma(t)\| \leq \tan\left[\tan^{-1}(\|\sigma_0\|) + \frac{\Lambda\|\omega_0\|(t+\tau)}{4}\right], \qquad t \in [-\tau, 0]$$

Additionally, since $t \in [-\tau, 0]$ and $\tau < \tau_{\max}$, we have the following upper bound for $\|\sigma\|$ over the initial condition interval:

$$\|\sigma(t)\| \leq \tan\left[\tan^{-1}(\|\sigma_0\|) + \frac{\Lambda\|\omega_0\|\tau_{\max}}{4}\right], \qquad t \in [-\tau, 0] \quad (57)$$

To stabilize the actual system, we formulate the Lipschitz-like condition from Eq. (33) for the perturbation term. Let $F_1(\sigma, q) = [B(\sigma) - I_{3\times 3}]q$ and $F_2(\sigma, q) = -4J^{-1}q^{\times}Jq$. To upper bound F_1, we need the following elegant property [14]:

$$B^T B = (1 + \sigma^2)^2 I_3 \quad (58)$$

(wherein we use the convenient notation $\sigma^2 \doteq \sigma^T\sigma$), which holds if $\|\sigma\|$ is bounded. Consider the term $(B - I_3)^T(B - I_3)$,

$$(B - I_3)^T(B - I_3) = B^T B - B - B^T + I_3$$

Substituting for $B^T B$, as well as for B in terms of σ from Eq. (3), and after simplifying, we obtain

$$(B - I_3)^T(B - I_3) = (\sigma^4 + 4\sigma^2)I_3 - 4\sigma\sigma^T I_3$$

Using the spectral shift property as well as the fact that eigenvalues of the outer product matrix $\sigma\sigma^T$ are 0, 0, and σ^2, we obtain

$$\|(B - I_3)\| = \max\{\sigma^4 + 4\sigma^2, \sigma^4 + 4\sigma^2, \sigma^4\} = \sigma^4 + 4\sigma^2 \quad (59)$$

Hence, $\|(B - I_3)\| = \|\sigma\|\sqrt{\sigma^2 + 4}$. Thus, we upper bound $F_1(\sigma, q)$ as

$$\|F_1(\sigma, q)\| \leq \|\sigma\|\sqrt{\sigma^2 + 4}\|q\|$$

Also, from Eq. (47), we have $F_2(\sigma, q) = -4J^{-1}q^{\times}Jq$, which implies $\|F_2(\sigma, q)\| \leq 4\Lambda^2\|q\|^2$. Combining the two terms, we obtain the following condition for the drift term:

$$\|F(\sigma, q)\| \leq \sqrt{(\sigma^2 + 4)\sigma^2 q^2 + 16\Lambda^4 q^4} \quad (60)$$

Now, if

$$\|F(\sigma, q)\|^2 < \gamma^2\|[\sigma^T, \omega^T]\|^2$$

is to be true, we need the following to be true, such that $t \geq 0$:

$$\|q\| < \frac{\gamma}{4\Lambda^2} \quad (61)$$

$$q^2(\sigma^2 + 4) < \gamma^2 \quad (62)$$

Eliminating $\|q\|$ from the second inequality leads to the constraint on σ as $\|\sigma\| < 2\sqrt{4\Lambda^4 - 1}$. Both these conditions can be satisfied if

$$\|\sigma, q\| < \min\left\{\frac{\gamma}{4\Lambda^2}, 2\sqrt{4\Lambda^4 - 1}\right\} \quad (63)$$

Equation (63) represents an upper bound for the state norm to be satisfied such that $t \geq 0$. Comparing with the generic result from Eq. (33), $\delta \doteq \min\{\gamma/(4\Lambda^2), 2\sqrt{4\Lambda^4 - 1}\}$. Enforcing Eq. (37) leads to

$$\|\sigma(t), q(t)\|_\tau < \sqrt{\frac{\alpha_1}{\alpha_2}}\min\left\{\frac{\gamma}{4\Lambda^2}, 2\sqrt{4\Lambda^4 - 1}\right\}, \qquad t \in [-\tau, 0] \quad (64)$$

Substituting for initial condition interval upper bounds on $\|\sigma(t)\|$ and $\|\omega(t)\|$ from Eqs. (55) and (57) in Eq. (64), we obtain the following region-of-attraction estimate condition:

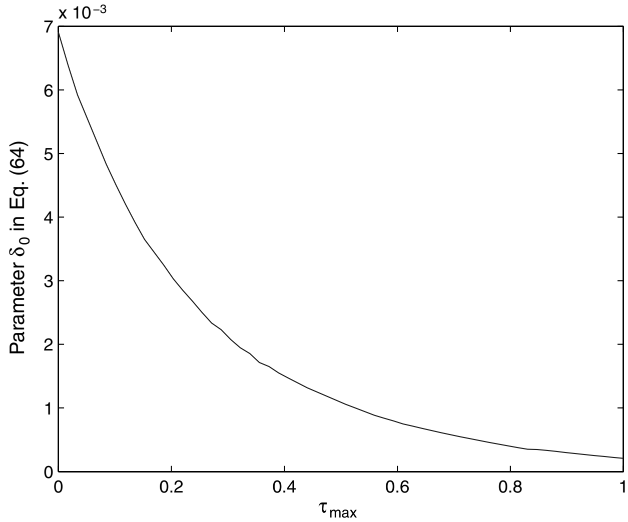

Fig. 3 δ_0 vs $\tau_{\max}$ comparison.

$$\tan^2\left[\tan^{-1}(\|\sigma_0\|) + \frac{\Lambda\|\omega_0\|\tau_{\max}}{4}\right] + \frac{\Lambda^2\omega_0^2}{16} < \frac{\alpha_1}{\alpha_2}\min\left\{\frac{\gamma^2}{16\Lambda^4}, 16\Lambda^4 - 4\right\} \doteq \delta_0^2 \quad (65)$$

Equation (65) is the region-of-attraction condition in terms of the state initial conditions σ_0 and ω_0 at time $-\tau$, where γ is evaluated in Eq. (50). Considering Eq. (65), the second term is found to be typically dominant over the first term. The region-of-attraction estimate was maximized through numerical optimization by choosing the free parameters W_0, W_1, W_2, ω_n, and ξ. We employ the inbuilt MATLAB function fmincon(). The performance index to be maximized is

$$\delta_0 = (\sqrt{\alpha_1/\alpha_2})\min\{\gamma/(4\Lambda^2), 2\sqrt{4\Lambda^4 - 1}\}$$

subject to constraints on control gains represented by Eq. (49) for exponential stability of the nominal system represented by three blocks of double integrators with delayed feedback. In doing so, we evaluate the analytical solution of the Lyapunov matrix. For a given region-of-attraction estimate, the permissible values of the initial attitude orientation state norm, as well as the initial angular velocity state norm, are precisely formulated by Eq. (50). Qualitatively, as the permissible initial attitude orientation increases, the permissible angular velocity state norm decreases. Figure 3 shows the dependence of δ_0 on $\tau_{\max}$, which again does not need information from the nonlinear perturbation.

V. Simulation Results

We implement the control design proposed in Sec. IV for the attitude stabilization problem with constant unknown delay in feedback. To generate realistic trajectories over the initial condition interval, the attitude dynamics are simulated torque free with initial conditions σ_0 and ω_0 over the delay interval $[-\tau, 0]$, which serves as the initial trajectory for the delay problem. J is chosen as diag(1000, 500, 700) from [2] for comparison; however, J is not required to be a diagonal matrix for our formulation. The quantity Λ turns out to be $\sqrt{2}$. The maximum time delay is chosen to be 0.0125 units. Using the numerical optimization process, we obtain δ_0 (where $\|\sigma, q\|_\tau < \delta_0$) to be 0.0065. In comparison with [2], the size of the region of attraction for the same (known) time delay is $\|\varphi\|_\tau \leq 0.0018$, where the state vector consists of the RP vector, angular velocity as well as an angular velocity filter. The size of the region of attraction considered in our work for a strict upper bound of 0.0125 on the unknown time delay is thus less conservative in comparison with the aforementioned result for a known time delay of 0.0125 [2]. The maximum principal rotation angle allowed in [2] for

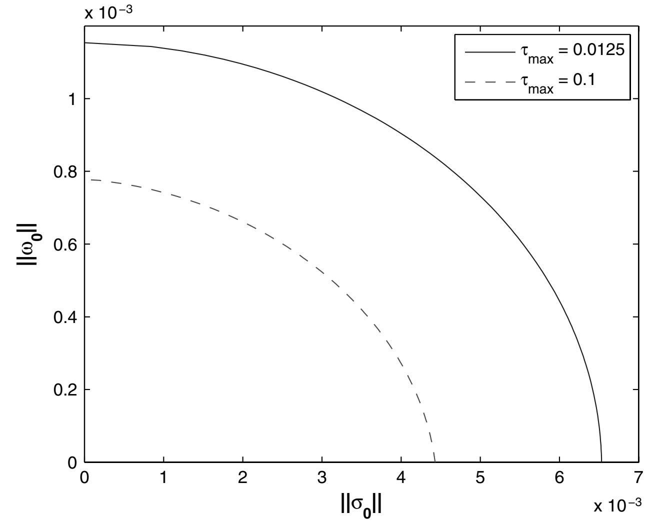

Fig. 4 Region of attraction: $\|\omega_0\|$ vs $\|\sigma_0\|$ for $\tau_{\max} = 0.0125$ and $\tau_{\max} = 0.1$.

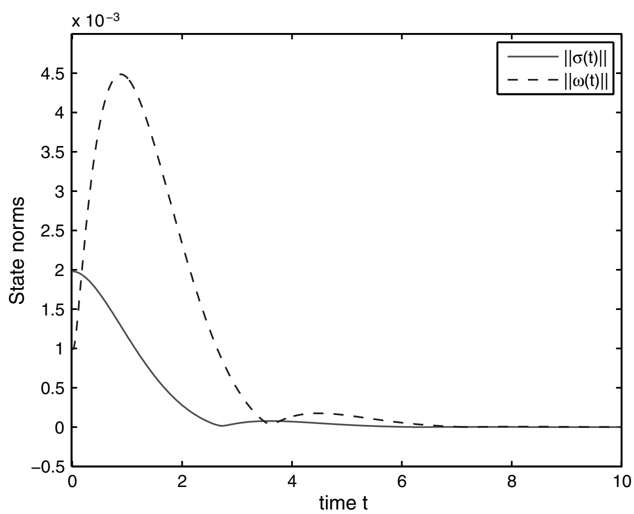

Fig. 5 Norms of MRP vector $\|\sigma(t)\|$ and angular rate $\|\omega(t)\|$ for $\tau_{\max} = 0.0125$ and $\tau = 0.012$.

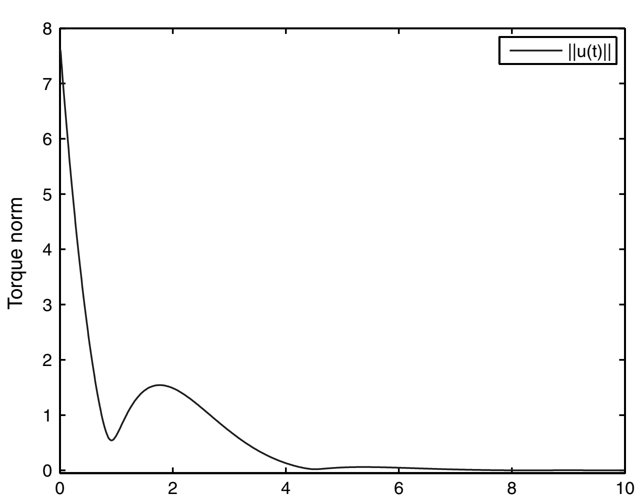

Fig. 6 Control torque norm $\|u(t)\|$ for $\tau_{\max} = 0.0125$ and $\tau = 0.012$.

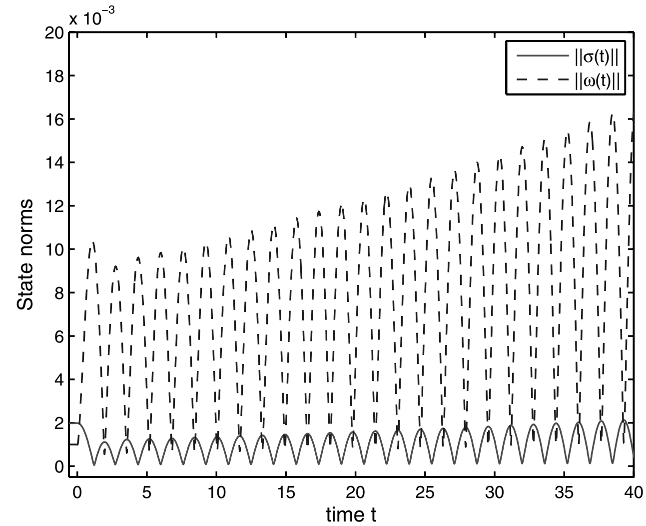

Fig. 7 Norms of MRP vector $\|\sigma(t)\|$ and angular rate $\|\omega(t)\|$ for $\tau_{\max} = 0.0125$ and $\tau = 0.6$.

$\tau = 0.0125$ is $2\tan^{-1}(0.0018) = 0.0036$ rad, which is approximately an order less than the maximum principal rotation angle of 0.0247 rad obtained in this work for the same unknown time delay.

In passing, we note that the estimate obtained using our approach is still potentially conservative. The gain parameters ξ and ω_n obtained using this process turn out to be $\xi = 0.7174$ and $\omega_n = 1.2386$. The W_i parameters turn out to be $W_0 = \mathrm{diag}(0.0786, 0.0790)$, $W_1 = \mathrm{diag}(0.0300, 0.0298)$, and $W_2 = \mathrm{diag}(0.1548, 0.1259)$. Figure 4 shows the estimate for the region of attraction in terms of initial state norms of σ_0 and ω_0 for $\tau_{\max} = 0.0125$ and for $\tau_{\max} = 0.1$. As is clearly observed, the region of attraction shrinks as the time delay increases. Reference [2] provides stability conditions for time delays strictly less than 0.04545.

For simulating the attitude dynamics equations with $\tau_{\max} = 0.0125$ and $\tau = 0.012$, ω_0 is chosen as $[0.001, \quad 0, \quad 0]^T$ and σ_0 is chosen as $[0, \quad 0.0014, \quad -0.0014]^T$. Figure 5 shows the trajectories of the state norms as a result of the implementation. As is observed, the state trajectories converge to the origin. Figure 6 shows the control history for the same simulation. To test the conservatism of the region-of-attraction estimate with respect to time delay, we choose the same initial conditions and controller parameters and increase the actual time-delay value τ. Instability is observed when the actual time delay is increased to $\tau = 0.6$. Figure 7 shows the first 40 s of the simulation. The reason for τ being relatively large for the onset of instability is the inherent conservatism arising from the complete-type L–K functional technique, which is well known [7].

VI. Conclusions

This paper considers the problem of finding an estimate of the region of attraction for rigid body attitude dynamics with an unknown time delay in feedback. The concept of the complete-type L–K functional is successfully extended in order to investigate stability for a class of nonlinear TDSs with unknown time delays. The modifications to the original complete L–K functional are critical toward ensuring that the region-of-attraction estimate is robust to the time delay. Exponential stability holds for all values of time delay less than the known upper bound. The region-of-attraction estimate is maximized through numerical optimization of the sufficient stability conditions by choosing the controller and other free parameters.

For application toward the attitude stabilization problem, the nominal system formulation in the form of double integrators ensures that the nonlinear term is a function of the current state value alone, permitting reduction in conservatism of the region-of-attraction estimate. For a given region-of-attraction estimate, the permissible values of the initial attitude orientation state norm, as well as the initial angular velocity state norm, are precisely formulated. As the permissible initial attitude orientation increases, the permissible angular velocity state norm decreases. The simulations verify the results obtained. It should be noted that the region-of-attraction estimate obtained is still somewhat conservative, even though simulations also show instability occurring if the actual time delay is higher than the upper bound considered. Although this work considers linear state feedback, future work could consider a controller with nonlinear state feedback in order to investigate whether the region-of-attraction estimate can be made less conservative. Future work could also include extending the analysis to unknown time-varying delay in feedback, extension of the control objective to achieve trajectory tracking and disturbance rejection, as well as extending the single spacecraft problem to achieving consensus in formation control with communication delay in feedback loops.

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